Problem Set 15 Solutions (Chapter 15)

Problem 15.1

Evaluate a Taylor expansion of equation (15.6) around $V = 0$.

Solution: Equation (15.6) from the document (page 3 [cite: 34]) relates the pair binding energy E to the interaction potential V:

$$\frac{1}{V} = \frac{N_F}{2} \log \left(\frac{2E_F + 2E_c - E}{2E_F - E} \right)$$

where N_F is the density of states at the Fermi energy E_F, and E_c is the cutoff energy. Let $\Delta E = 2E_F - E$ be the binding energy (the amount E is below the energy of two non-interacting electrons at the Fermi level). The equation becomes:

$$\frac{1}{V} = \frac{N_F}{2} \log \left(\frac{\Delta E + 2E_c}{\Delta E} \right) = \frac{N_F}{2} \log \left(1 + \frac{2E_c}{\Delta E} \right)$$

Rearranging for ΔE:

$$\frac{2}{N_F V} = \log \left(1 + \frac{2E_c}{\Delta E} \right)$$

$$e^{2/(N_F V)} = 1 + \frac{2E_c}{\Delta E}$$

$$\Delta E = \frac{2E_c}{e^{2/(N_F V)} - 1}$$

We are asked for an expansion around $V = 0$. Let's consider the behavior for small positive V. In this case, $V \to 0^+$, the exponent $2/(N_F V)$ becomes very large and positive. Therefore, $e^{2/(N_F V)} \gg 1$. The expression for ΔE can be approximated as:

$$\Delta E \approx \frac{2E_c}{e^{2/(N_F V)}} = 2E_c e^{-2/(N_F V)}$$

Substituting back $E = 2E_F - \Delta E$:

$$E \approx 2E_F - 2E_c e^{-2/(N_F V)}$$

This expression shows the energy E for small V. It demonstrates that for any attractive potential ($V > 0$), no matter how small, there is a bound state ($E < 2E_F$). A direct Taylor series expansion in powers of V around $V = 0$ is not possible because the function $e^{-k/V}$ has an essential singularity at $V = 0$. The derived expression $E \approx 2E_F - 2E_c e^{-2/(N_F V)}$ is the appropriate description for small V, showing the non-analytic dependence on the coupling constant V. This result itself is derived on page 3[cite: 34].

Problem 15.2

Problem 8.5 showed that for a Kibble balance the current I measured in the dynamic phase and the voltage V measured in the static phase are related to the mass m, gravitational constant g, and velocity v by $IV = mgv$. Using the inverse AC Josephson effect (equation 15.25) to determine the voltage, and the quantum Hall effect (equation 14.41) along with the inverse AC Josephson effect to determine the current, relate the measurement to fundamental constant(s).

Solution: The Kibble balance equation is given as $IV = mgv$[cite: 268]. The voltage V is measured using the inverse AC Josephson effect (page 6, Eq. 15.25 [cite: 65, 268]):

$$V = n \frac{h}{2e} f$$

where n is an integer, h is Planck's constant, e is the elementary charge, and f is the frequency of the microwaves applied to the Josephson junction array.

The current I is measured by comparing the voltage drop $V_R = IR$ across a standard resistor R to the Josephson voltage standard. The resistance R itself is calibrated using the quantum Hall effect. The quantum Hall resistance is given by (from Chapter 14, Eq. 14.41, cited on page 22 [cite: 268]):

$$R_H = \frac{1}{i} \frac{h}{e^2}$$

where i is an integer (or fraction in FQHE). A standard resistor R is calibrated against R_H, so $R = rR_H = \frac{r}{i}\frac{h}{e^2}$, where r is a calibration ratio (ideally an integer or simple rational number for precision). The voltage drop $V_R = IR$ is measured by comparison with a Josephson voltage:

$$V_R = n'\frac{h}{2e}f'$$

where n' is another integer and f' is another frequency. Therefore,

$$I = \frac{V_R}{R} = \frac{n'\frac{h}{2e}f'}{\frac{r}{i}\frac{h}{e^2}} = \frac{n'i}{2r}\frac{h}{e}f'\frac{e^2}{h} = \frac{n'i}{2r}ef'$$

Now substitute the expressions for I and V into the Kibble balance equation $IV = mgv$:

$$\left(\frac{n'i}{2r}ef'\right)\left(n\frac{h}{2e}f\right) = mgv$$

$$\frac{nn'i}{4r}hff' = mgv$$

This equation relates the macroscopic measurement mgv to the fundamental constant h (Planck's constant) and experimentally controlled frequencies (f, f') and velocity (v), along with integers (n, n', i) and the calibration factor (r). Since e cancels out, the measurement primarily links mass to Planck's constant. This relationship forms the basis for the redefinition of the kilogram by fixing the value of h.

Problem 15.3

If a SQUID with an area of $A = 1$ cm^2 can detect 1 flux quantum, how far away can it sense the field from a wire carrying 1 A?

Solution: A SQUID (Superconducting Quantum Interference Device) can detect extremely small magnetic fluxes. The quantum of magnetic flux is given by Equation (15.27) (page 6 [cite: 73]):

$$\Phi_0 = \frac{hc}{2e} = 2.07 \times 10^{-7}\,\mathrm{G \cdot cm^2}$$

We need to convert this to SI units (Tesla meter^2, or Weber): $1\,\mathrm{T} = 10^4\,\mathrm{G}$ and $1\,\mathrm{m}^2 = 10^4\,\mathrm{cm}^2$.

$$\Phi_0 = (2.07 \times 10^{-7}\,\mathrm{G \cdot cm^2}) \times \frac{1\,\mathrm{T}}{10^4\,\mathrm{G}} \times \frac{1\,\mathrm{m}^2}{10^4\,\mathrm{cm}^2} = 2.07 \times 10^{-15}\,\mathrm{T \cdot m^2} = 2.07 \times 10^{-15}\,\mathrm{Wb}$$

The SQUID has an area $A = 1\,\mathrm{cm}^2 = 1 \times 10^{-4}\,\mathrm{m}^2$[cite: 269]. The minimum detectable flux is $\Phi_{min} = 1 \times \Phi_0 = 2.07 \times 10^{-15}\,\mathrm{Wb}$. Assuming the magnetic field B is perpendicular to the SQUID loop area A, the minimum detectable magnetic field B_{min} is:

$$B_{min} = \frac{\Phi_{min}}{A} = \frac{2.07 \times 10^{-15}\,\mathrm{T \cdot m^2}}{1 \times 10^{-4}\,\mathrm{m}^2} = 2.07 \times 10^{-11}\,\mathrm{T}$$

The magnetic field B produced by a long straight wire carrying a current I at a distance r is given by Ampere's Law:

$$B = \frac{\mu_0 I}{2\pi r}$$

where $\mu_0 = 4\pi \times 10^{-7}$ T·m/A is the permeability of free space. We want to find the distance r at which the field from a wire carrying $I = 1$ A equals B_{min}:

$$B_{min} = \frac{\mu_0 I}{2\pi r}$$

$$r = \frac{\mu_0 I}{2\pi B_{min}}$$

$$r = \frac{(4\pi \times 10^{-7}\,\mathrm{T \cdot m/A}) \times (1\,\mathrm{A})}{2\pi \times (2.07 \times 10^{-11}\,\mathrm{T})}$$

$$r = \frac{2 \times 10^{-7}}{}$$

$\mathrm{m} \approx 9662\,\mathrm{m} So, the SQUID can sense the field from a 1 A wire from a distance of approximately 9.7\,\mathrm{km}.$

Problem 15.4

Typical parameters for a quartz resonator are $C_e = 5pF$, $C_m = 20fF$, $L_m = 3mH$, $R_m = 6\Omega$. Plot, and explain, the dependence of the reactance (imaginary part of the impedance), resistance (real part), and the phase angle of the impedance on the frequency.

Solution: The equivalent circuit for the piezoelectric resonator is given in Figure 15.4 (page 14 [cite: 150]). It consists of a static capacitance C_e in parallel with a motional arm comprising R_m, L_m, and C_m in series. The component values are $C_e = 5\,\text{pF}$, $C_m = 20\,\text{fF}$, $L_m = 3\,\text{mH}$, and $R_m = 6\,\Omega$[cite: 270].

The impedance of the motional arm (Z_m) and the electrical capacitance (Z_e) are:

$$Z_m = R_m + j\left(\omega L_m - \frac{1}{\omega C_m}\right) = R_m + jX_m$$

$$Z_e = \frac{1}{j\omega C_e} = -j\frac{1}{\omega C_e}$$

The total impedance Z_{total} is the parallel combination:

$$Z_{total} = \frac{Z_m Z_e}{Z_m + Z_e} = \frac{(R_m + jX_m)(-j/(\omega C_e))}{R_m + j(X_m - 1/(\omega C_e))}$$

We can analyze the behavior around the series resonance frequency ω_s and the parallel resonance (antiresonance) frequency ω_p.

1. **Series Resonance Frequency (ω_s):** This occurs when the reactance of the motional arm is zero ($X_m = 0$).

$$\omega_s = \frac{1}{\sqrt{L_m C_m}} = \frac{1}{\sqrt{(3\times10^{-3}\,\text{H})(20\times10^{-15}\,\text{F})}} = \frac{1}{\sqrt{60\times10^{-18}\,\text{s}^2}} \approx 1.291\times10^{8}\,\text{rad/s}$$

$$f_s = \frac{\omega_s}{2\pi} \approx 20.55\,\text{MHz}$$

At ω_s, $Z_m = R_m = 6\,\Omega$. The total impedance is $Z_{total}(\omega_s) = \frac{R_m Z_e}{R_m + Z_e}$. Since R_m is very small compared to $|Z_e|$ at this frequency ($|Z_e| = 1/(\omega_s C_e) \approx 1540\,\Omega$), the total impedance $Z_{total}(\omega_s)$ is very low, close to R_m, and slightly capacitive. The resistance $R = \text{Re}(Z_{total})$ reaches its minimum value near ω_s. The reactance $X = \text{Im}(Z_{total})$ is small and negative. The phase angle $\phi = \arctan(X/R)$ is close to $0°$ (slightly negative).

2. **Parallel Resonance Frequency (ω_p):** This occurs when the total impedance is maximum (ideally infinite for $R_m = 0$). It happens slightly above ω_s. The approximate frequency is given by $\omega_p \approx \omega_s\sqrt{1 + C_m/C_e}$.

$$\frac{C_m}{C_e} = \frac{20\times10^{-15}\,\text{F}}{5\times10^{-12}\,\text{F}} = 0.004$$

$$\omega_p \approx \omega_s\sqrt{1+0.004} \approx 1.002\,\omega_s \approx 1.294\times10^{8}\,\text{rad/s}$$

$$f_p = \frac{\omega_p}{2\pi} \approx 20.59\,\text{MHz}$$

Near ω_p, the reactance of the inductive motional arm ($X_m > 0$) nearly cancels the reactance of C_e. The total impedance $Z_{total}(\omega_p)$ becomes very high and largely resistive. The resistance R reaches its maximum value near ω_p. The reactance X crosses zero from positive to negative at ω_p. The phase angle ϕ passes through $0°$ at ω_p.

Frequency Dependence Summary:

- **Resistance ($R = \textbf{Re}(Z_{total})$):** Starts high at low frequencies (dominated by C_e), drops to a sharp minimum near ω_s (value close to R_m), rises rapidly to a very sharp maximum near ω_p, and then decreases again, approaching zero at high frequencies.

- **Reactance ($X = \textbf{Im}(Z_{total})$):** Starts large and negative (capacitive) at low frequencies (dominated by C_e). Becomes less negative, passes through a small negative value near ω_s. Rises sharply, becoming large and positive (inductive) between ω_s and ω_p. Crosses zero at ω_p. Becomes large and negative (capacitive) again above ω_p, approaching the reactance of C_e at very high frequencies.

- **Phase Angle ($\phi = \arctan(X/R)$):** Starts near $-90°$ at low frequencies. Rises towards $0°$ near ω_s. Quickly rises to near $+90°$ between ω_s and ω_p. Passes through $0°$ at ω_p. Drops rapidly back towards $-90°$ above ω_p.

The region between f_s and f_p is characterized by inductive behavior and rapidly changing phase, which is exploited in oscillator circuits[cite: 154]. The high quality factor $Q = (\omega_s L_m)/R_m \approx (1.29 \times 10^8\,\text{rad/s} \times 3 \times 10^{-3}\,\text{H})/6\,\Omega \approx 64500$ ensures the resonances are extremely sharp.

Problem 15.5

If a ship traveling on the equator uses one of John Harrison's chronometers to navigate, what is the error in its position after one month? What if it uses a cesium beam atomic clock?

Solution: Longitude determination requires accurate timekeeping. An error in time Δt leads to an error in longitude (position) Δx. For a ship on the equator:

- Earth's equatorial radius $R_E \approx 6378\,\text{km}$.
- Equatorial circumference $C = 2\pi R_E \approx 40\,075\,\text{km}$.
- Earth's rotational period $T = 1\,\text{day} = 86\,400\,\text{s}$.
- Speed of a point on the equator $v = C/T = 40\,075\,\text{km}/86\,400\,\text{s} \approx 0.4638\,\text{km/s} = 463.8\,\text{m/s}$.
- Position error $\Delta x = v \times \Delta t$.

We need to calculate the total time error Δt after one month (approximately 30 days) for each clock.

1. **John Harrison's Chronometer:** The text mentions Harrison's chronometer was good to better than $1\,\text{s/day}$[cite: 221]. Let's assume an error rate of $\delta t_H/\text{day} = 1\,\text{s/day}$. Total time error over 30 days:

$$\Delta t_H = (1\,\text{s/day}) \times 30\,\text{days} = 30\,\text{s}$$

Position error:

$$\Delta x_H = v \times \Delta t_H = (463.8\,\text{m/s}) \times (30\,\text{s}) = 13\,914\,\text{m} \approx 13.9\,\text{km}$$

The position error after one month using Harrison's chronometer could be around 14 km.

2. **Cesium Beam Atomic Clock:** The relative uncertainty (instability) of a cesium beam clock is $\delta t/t \approx 10^{-12}$[cite: 198]. Duration $t = 30\,\text{days} = 30 \times 86\,400\,\text{s} = 2.592 \times 10^6\,\text{s}$. Total time error over 30 days:

$$\Delta t_{Cs} = (\delta t/t) \times t = 10^{-12} \times (2.592 \times 10^6\,\text{s}) = 2.592 \times 10^{-6}\,\text{s}$$

Position error:

$$\Delta x_{Cs} = v \times \Delta t_{Cs} = (463.8\,\text{m/s}) \times (2.592 \times 10^{-6}\,\text{s}) \approx 1.20 \times 10^{-3}\,\text{m} = 1.2\,\text{mm}$$

The position error after one month using a cesium atomic clock is negligible for navigation purposes, about 1.2 mm.

Problem 15.6

GPS satellites orbit at an altitude of 20 180 km. (a) How fast do they travel? (b) What is their orbital period? (c) Estimate the special-relativistic correction over one orbit between a clock on a GPS satellite and one on the Earth. Which clock goes slower? (d) What is the general-relativistic correction over one orbit? Which clock goes slower?

Solution: Given:

- Altitude $h = 20\,180\,\text{km}$ [cite: 272]
- Earth's radius $R_E \approx 6378\,\text{km}$
- Orbital radius $r = R_E + h = 6378\,\text{km} + 20\,180\,\text{km} = 26\,558\,\text{km} = 2.6558 \times 10^7\,\text{m}$
- Earth's mass $M_E = 5.972 \times 10^{24}\,\text{kg}$
- Gravitational constant $G = 6.674 \times 10^{-11}\,\text{N} \cdot \text{m}^2/\text{kg}^2$
- Speed of light $c = 3 \times 10^8\,\text{m/s}$

(a) **Satellite Speed (v):** For a circular orbit, gravity provides the centripetal force: $\frac{GM_E m}{r^2} = \frac{mv^2}{r}$.

$$v^2 = \frac{GM_E}{r} = \frac{(6.674 \times 10^{-11}\,\mathrm{N \cdot m^2/kg^2})(5.972 \times 10^{24}\,\mathrm{kg})}{2.6558 \times 10^7\,\mathrm{m}} \approx 1.500 \times 10^7\,\mathrm{m^2/s^2}$$

$$v = \sqrt{1.500 \times 10^7\,\mathrm{m^2/s^2}} \approx 3873\,\mathrm{m/s} \approx 3.87\,\mathrm{km/s}$$

The satellites travel at approximately 3.87 km/s[cite: 273].

(b) **Orbital Period (T):**

$$T = \frac{2\pi r}{v} = \frac{2\pi(2.6558 \times 10^7\,\mathrm{m})}{3873\,\mathrm{m/s}} \approx 43\,075\,\mathrm{s}$$

Converting to hours: $T \approx 43\,075\,\mathrm{s}/(3600\,\mathrm{s/hr}) \approx 11.97$ hours. This is approximately half a sidereal day, the actual orbital period for GPS satellites[cite: 273].

(c) **Special Relativistic Correction:** Due to its velocity, the satellite's clock experiences time dilation relative to a stationary clock at the same gravitational potential. The satellite clock runs slower (Equation 15.55 [cite: 206]). The fractional difference in clock rate is $\Delta f/f \approx -\frac{1}{2}\frac{v^2}{c^2}$.

$$\frac{v^2}{c^2} = \frac{1.500 \times 10^7\,\mathrm{m^2/s^2}}{(3 \times 10^8\,\mathrm{m/s})^2} = \frac{1.500 \times 10^7}{}$$

$\approx 1.667{\times}10^{-10}$ $\frac{\Delta f}{f} \approx -\frac{1}{2}(1.667{\times}10^{-10}) \approx -8.335{\times}10^{-11}$ $Thenegativesignindicatesthesatelliteclockisslower.Timed$
$= 43\,075\,\mathrm{s})$:$\Delta t_{SR} = \left|\frac{\Delta f}{f}\right| {\times} T = (8.335{\times}10^{-11}){\times}(43\,075\,\mathrm{s}) \approx 3.59{\times}10^{-6}\,\mathrm{s} = 3.59\,\mathrm{\mu s}$ $Duetospecialrelativity, thesatelliteclo$
275].

(d) **General Relativistic Correction:** Clocks run slower in stronger gravitational fields (closer to Earth). The satellite is in a weaker field than a clock on Earth's surface. Therefore, the satellite clock runs faster due to general relativity (Equation 15.59 [cite: 215]). The fractional frequency difference due to gravitational potential $\phi = -GM/r$ is $\Delta f/f \approx -(\phi_{sat} - \phi_{Earth})/c^2 = (GM_E/(R_E c^2)) - (GM_E/(rc^2))$.

$$\frac{GM_E}{R_E c^2} = \frac{(6.674 \times 10^{-11})(5.972 \times 10^{24})}{(6.378 \times 10^6)(3 \times 10^8)^2} \approx 6.95 \times 10^{-10}$$

$\frac{GM_E}{rc^2} = \frac{(6.674{\times}10^{-11})(5.972{\times}10^{24})}{(2.6558{\times}10^7)(3{\times}10^8)^2} \approx 1.67 \times 10^{-10}$ $Fractionaldifference$:$\frac{\Delta f}{f} \approx 6.95 \times 10^{-10} - 1.67 \times 10^{-10} = 5.28{\times}10^{-10}$ $Thepositivesignindicatesthesatelliteclockrunsfaster.Timedifferenceoveroneorbit$(T
$= 43\,075\,\mathrm{s})$:$\Delta t_{GR} = \left(\frac{\Delta f}{f}\right) {\times} T = (5.28{\times}10^{-10}){\times}(43\,075\,\mathrm{s}) \approx 2.27{\times}10^{-5}\,\mathrm{s} = 22.7\,\mathrm{\mu s}$ $Duetogeneralrelativity, thesatellitec$
275].

Net Effect: The general relativistic effect (faster clock) dominates the special relativistic effect (slower clock). The net effect is that the satellite clock runs faster than an Earth-based clock by approximately $22.7\,\mathrm{\mu s} - 3.59\,\mathrm{\mu s} \approx 19.1\,\mathrm{\mu s}$ per orbit. This corresponds to about 38 µs per day, a correction essential for GPS accuracy[cite: 218, 247].